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Lie algebras under constraints and non-bijective canonical transformations

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Abstract. The concept of a Lie algebra under constraints is developed in connection with the theory of non-bijective canonical transformations. A finite-dimensional vector space M, carrying a faithful linear representation of a Lie algebra L, is mapped into a lower-dimensional space $\tilde{M}$ in such a manner that a subalgebra L_0 of L is mapped into $D(L_0)=0$. The Lie algebra L under the constraint $D(L_0)=0$ is the largest subalgebra L_1 of L that can be represented faithfully on $\tilde{M}$. If L_0 is semisimple, then L_1 is shown to be the centraliser $\mathrm{cent}_L L_0$. If L is semisimple and L_0 is a one-dimensional diagonal subalgebra of a Cartan subalgebra of L, then L_1 is shown to be the factor algebra $\mathrm{cent}_L L_0/L_0$. The latter two results are applied to non-bijective canonical transformations generalising the Kustaanheimo-Stiefel transformation.

1. Introduction

In the recent years, the LC transformation (Levi-Civita 1956), an $R^2 \to R^2$ map with discrete kernel, and the KS transformation (Kustaanheimo and Stiefel 1965), an $R^4 \to R^3$ map with continuous kernel, have been investigated and used in various domains of theoretical physics. The LC transformation is closely related to the usual conformal map and is therefore connected to the algebra of ordinary complex numbers. The KS transformation may be considered as a byproduct of the theory of spinors and thus turns out to be connected to the algebra of ordinary quaternions (Kustaanheimo and Stiefel 1965, Blanchard and Sirugue 1981, Vivarelli 1983, Cornish 1984, Kibler and Négadi 1984b). The LC and KS transformations have been employed in classical and quantum mechanics and the reader is referred to the paper by Lambert and Kibler (1988) for an extensive bibliography. Let us just mention that the KS transformation is of interest in the study of dynamical systems either in a partial-differential-equation approach (Ikeda and Miyachi 1970, Boiteux 1972, Barut *et al* 1979, Kibler and Négadi 1984b) or in a path-integral approach (Duru and Kleinert 1979, Blanchard and Sirugue 1981, Young and DeWitt-Morette 1986) or in a phase-space approach (Gracia-Bondía 1984). In this direction, the KS transformation has been very recently applied to a quantum mechanical investigation of the Hartmann potential (Kibler and Winternitz 1987) and of a Aharonov–Bohm-like potential (Kibler and Négadi 1987).

There exist several non-bijective quadratic transformations generalising the LC and KS transformations. In particular, Kibler and Négadi (1984b) (see also Lambert and Kibler 1988) have studied a compact $R^4 \to R^4$ transformation and a compact $R^2 \to R^+$

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transformation that parallel the LC and KS transformations, respectively. Furthermore, Iwai (1985) and Iwai and Rew (1985) have defined and used in symmetry reduction problems an $R^4 \to R^3$ transformation which may be thought of as a non-compact extension of the KS transformation. General attempts to introduce non-bijective quadratic transformations have been achieved by Boiteux (1982), Polubarinov (1984) and Lambert *et al* (1986). Finally, Lambert and Kibler (1987, 1988) have recently introduced and studied from both an algebraic and geometric viewpoint both (i) compact and non-compact $R^{2m} \to R^{2m}$ transformations with $2m = 2, 4, 8, \ldots$ extending the LC transformation and referred to as quasi-Hurwitz transformations and (ii) compact and non-compact $R^{2m} \to R^{2m-n}$ transformations with $(2m, 2m-n) = (2, 1)$, $(4, 1)$, $(4, 3)$, $(8, 1)$ and $(8, 5)$ extending the KS transformation and referred to as Hurwitz transformations. Such a study is based on the use of anti-involutions of Cayley-Dickson algebras, the latter algebras being generalisations of the algebras of complex numbers, quaternions and octonions.

It is the aim of this paper to develop a group theoretical approach to the Hurwitz transformations $R^{2m} \to R^{2m-n}$, with $(2m, 2m-n) = (2, 1)$, $(4, 3)$ and $(8, 5)$, which comprise and extend the KS transformation. The whole philosophy of this approach may be summed up as follows. In view of the non-bijectivity of the $R^{2m} \to R^{2m-n}$ map, we may introduce, for $2m$ fixed, $n = m - 1 + \delta(m, 1)$ 1-forms which are not total differentials and equate them to zero. We can then associate a vector field to each of the n 1-forms arising in the $R^{2m} \to R^{2m-n}$ transformation. For $2m$ fixed, each of the vector fields X_i with $i = 1, 2, \ldots, n$ is defined in the real symplectic Lie algebra $\mathrm{sp}(4m, R)$ and the n vector fields together span a subalgebra L_0 of $\mathrm{sp}(4m, R)$. Indeed, the algebra L_0 may be considered as a specific realisation of the Lie algebra of the ambiguity group discussed by Mello and Moshinsky (1975) and Moshinsky and Seligman (1978, 1979) in connection with general $R^p \to R^{p'}$ (non-bijective) transformations with $p < p'$. The algebra L_0 will be called a *constraint Lie algebra* since its n generators X_i satisfy $X_i\psi = 0$ for any function ψ of class $C(R^{2m-n})$. (In this vein, it is to be noted that the constraints $X_i = 0$ $(i = 1, 2, \ldots, n)$ written in the phase space $R^{2m} \times R^{2m}$ are nothing but primary constraints of the generalised Hamiltonian formalism developed by Dirac (1964).) At this stage, one may ask the question: what is the group theoretical significance of the constraint conditions (also called superselection rules by Boiteux (1982)) $X_i\psi = 0$, $i = 1, 2, \ldots, n$? In other words, what is the subalgebra of $\mathrm{sp}(4m, R)$ which survives when one forces the generators of $\mathrm{L}_0 \subset \mathrm{sp}(4m, R)$ to vanish? These questions lead to studying *Lie algebras under constraints* and this is done in the present paper by introducing various constraints in $\mathrm{sp}(4m, R)$ for $2m = 2$, 4 and 8.

This article constitutes a non-trivial extension of a series of papers by Kibler and Négadi (1983a, b, 1984a). In the latter works a unique constraint $X = 0$, corresponding to a constraint Lie algebra L_0 of type $\mathrm{so}(2)$ for the KS transformation, is introduced into $\mathrm{sp}(8, R)$. This leads to a Lie algebra under constraints isomorphic to $\mathrm{so}(4, 2)$. As a physical application, the non-invariance dynamical algebra $\mathrm{so}(4, 2)$ of the R^3 hydrogen atom may be obtained from the non-invariance dynamical algebra $\mathrm{sp}(8, R)$ of the R^4 isotropic harmonic oscillator. This important result is a group theoretical complement of the well known result that the KS transformation allows us to convert, in a Schrödinger, Feynman or Weyl-Wigner-Moyal formulation, the R^3 hydrogen atom problem into the R^4 isotropic harmonic oscillator problem. The $\mathrm{sp}(8, R)$-$\mathrm{so}(4, 2)$ connection has been further worked out (i) by Quesne (1986) in relation to the independent-electron dynamical group of intrashell many-electron states as well as with the correlated electron dynamical group of intrashell doubly excited states and

(ii) by Georgieva *et al* (1986) in relation to boson representations of symplectic algebras and their application to the theory of nuclear structure.

The Hurwitz transformations generalising the KS transformation are described in § 2 in a unified and original way. Although the material contained in § 2 turns out to be a byproduct of the work by Lambert and Kibler (1987, 1988), the adopted presentation is self-consistent and constitutes an alternative to the derivation of the Hurwitz transformations. Some general results on Lie algebras under constraints are presented as theorems in § 3, where they are also applied to the Hurwitz transformations of § 2. Constraint subalgebras L_0 of symplectic Lie algebras L are investigated in § 4 for cases where L_0 and L are more general than for the cases corresponding to the Hurwitz transformations of § 2. The final § 5 is devoted to some concluding remarks.

2. Hurwitz transformations

We start from the (generalised) Hurwitz matrix

$$A(\boldsymbol{u})=\begin{bmatrix} -u_0 & c_1u_1 & c_2u_2 & -c_1c_2u_3 & c_3u_4 & -c_1c_3u_5 & -c_2c_3u_6 & c_1c_2c_3u_7 \\ u_1 & -u_0 & c_2u_3 & -c_2u_2 & c_3u_5 & -c_3u_4 & c_2c_3u_7 & -c_2c_3u_6 \\ u_2 & -c_1u_3 & -u_0 & c_1u_1 & c_3u_6 & -c_1c_3u_7 & -c_3u_4 & c_1c_3u_5 \\ u_3 & -u_2 & u_1 & -u_0 & c_3u_7 & -c_3u_6 & c_3u_5 & -c_3u_4 \\ u_4 & -c_1u_5 & -c_2u_6 & c_1c_2u_7 & -u_0 & c_1u_1 & c_2u_2 & -c_1c_2u_3 \\ u_5 & -u_4 & -c_2u_7 & c_2u_6 & u_1 & -u_0 & -c_2u_3 & c_2u_2 \\ u_6 & c_1u_7 & -u_4 & -c_1u_5 & u_2 & c_1u_3 & -u_0 & -c_1u_1 \\ u_7 & u_6 & -u_5 & -u_4 & u_3 & u_2 & -u_1 & -u_0 \end{bmatrix} \tag{1}$$

in dimension $2m=8$, where $u_\alpha (\alpha=0,1,\ldots,7)$ are real numbers and $c_k=\pm1$ $(k=1,2,3)$. We also consider the column vector $\boldsymbol{u}$, the metric matrix η and the conjugation matrix ε defined by (the sign $\sim$ indicating matrix transposition):

$$\begin{aligned} \tilde{\boldsymbol{u}} &= (-u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7) \\ \eta &= \operatorname{diag}(1, -c_1, -c_2, c_1c_2, -c_3, c_1c_3, c_2c_3, -c_1c_2c_3) \\ \varepsilon &= \operatorname{diag}(1, -1, 1, 1, 1, 1, -1, -1). \end{aligned} \tag{2}$$

Let us finally introduce the matrices $A(\boldsymbol{u})$, $\boldsymbol{u}$, η and ε in dimensions $2m=4$ and 2 in the following manner: $A(\boldsymbol{u})$, η and ε are the $2m\times2m$ matrices consisting of the first $2m$ rows and columns of the corresponding matrices defined by equations (1) and (2), whereas $\boldsymbol{u}$ is the column vector consisting of the first $2m$ rows of the corresponding column vector defined by equation (2).

It can be verified that the matrices $A(\boldsymbol{u})$ for $2m=2$, 4 and 8 satisfy the properties

$$\tilde{A}(\boldsymbol{u})\eta A(\boldsymbol{u})=(\tilde{\boldsymbol{u}}\eta\boldsymbol{u})\eta \qquad \tilde{A}(\boldsymbol{u})=\eta[-A(\boldsymbol{u})-2u_0I_{2m}]\eta \tag{3}$$

where I_{2m} stands for the $2m\times2m$ unit matrix. The matrices $A(\boldsymbol{u})$ are of central importance in the celebrated Hurwitz (1898) theorem of arithmetics and its non-compact extension (Lambert and Kibler 1988). (The compact cases treated by Hurwitz correspond to $c_1=c_2=c_3=-1$, $c_1=c_2=-1$ and $c_1=-1$ for $2m=8$, 4 and 2, respectively.) The matrices $A(\boldsymbol{u})$ in dimensions $2m=2$, 4 and 8 are related to the Cayley-Dickson algebras $A(c_1)$, $A(c_1,c_2)$ and $A(c_1,c_2,c_3)$ of dimensions $2m=2$, 4 and 8 and they may be written in terms of elements of Clifford algebras of degrees $2m-1=1$, 3 and 7, respectively (Lambert and Kibler 1988). In this respect, in the compact case

$c_1 = c_2 = c_3 = -1$ for $2m = 8$, the Clifford algebra of degree $2m - 1 = 7$ has been recently considered by Shaw (1988) in connection with a new view of the $d = 7$ Dirac algebra.

We are now in a position to define non-bijective quadratic transformations. We shall deal in turn with the cases $2m = 8$, 4 and 2.

2.1. *The case 2m = 8*

Let us define the $R^8 \to R^5$ map through

$$x = A(u)\varepsilon u. \tag{4}$$

In detail, we have

$$\begin{aligned} x_0 &= u_0^2 - c_1 u_1^2 + c_2 u_2^2 - c_1 c_2 u_3^2 + c_3 u_4^2 - c_1 c_3 u_5^2 + c_2 c_3 u_6^2 - c_1 c_2 c_3 u_7^2 \\ x_2 &= 2(-u_0 u_2 + c_1 u_1 u_3 + c_3 u_4 u_6 - c_1 c_3 u_5 u_7) \\ x_3 &= 2(-u_0 u_3 + u_1 u_2 - c_3 u_5 u_6 + c_3 u_4 u_7) \\ x_4 &= 2(-u_0 u_4 + c_1 u_1 u_5 - c_2 u_2 u_6 + c_1 c_2 u_3 u_7) \\ x_5 &= 2(-u_0 u_5 + u_1 u_4 + c_2 u_3 u_6 - c_2 u_2 u_7). \end{aligned} \tag{5}$$

(In the general algebraic framework developed by Lambert and Kibler (1988), the $R^8 \to R^5$ map given by (5) corresponds to the right Hurwitz transformation $\mathscr{K}_{\mathrm{R}}^{(7)}$ associated to the anti-involution j_7 of $A(c_1, c_2, c_3)$.) Equation (5) may equally well be seen to result from the integration of $2A(u)\varepsilon\, du$. Indeed, the column vector $2A(u)\varepsilon\, du$ is the transpose of the row vector $(dx_0, \omega_1, dx_2, dx_3, dx_4, dx_5, \omega_6, \omega_7)$, where the 1-forms

$$\begin{aligned} \omega_1 &= 2(-u_1 du_0 + u_0 du_1 + c_2 u_3 du_2 - c_2 u_2 du_3 + c_3 u_5 du_4 - c_3 u_4 du_5 - c_2 c_3 u_7 du_6 + c_2 c_3 u_6 du_7) \\ \omega_6 &= 2(-u_6 du_0 - c_1 u_7 du_1 - u_4 du_2 - c_1 u_5 du_3 + u_2 du_4 + c_1 u_3 du_5 + u_0 du_6 + c_1 u_1 du_7) \\ \omega_7 &= 2(-u_7 du_0 - u_6 du_1 - u_5 du_2 - u_4 du_3 + u_3 du_4 + u_2 du_5 + u_1 du_6 + u_0 du_7) \end{aligned} \tag{6}$$

which are not total differentials, can be taken to be equal to zero in view of the non-bijectivity of the $R^8 \to R^5$ map. The constraints $\omega_1 = \omega_6 = \omega_7 = 0$ make it possible to obtain

$$dx_0^2 - c_2\, dx_2^2 + c_1 c_2\, dx_3^2 - c_3\, dx_4^2 + c_1 c_3\, dx_5^2 = 4r(d\tilde{u}\, \eta\, du) \tag{7}$$

where the 'distance' $r = \tilde{u}\, \eta\, u$ satisfies

$$r^2 = x_0^2 - c_2 x_2^2 + c_1 c_2 x_3^2 - c_3 x_4^2 + c_1 c_3 x_5^2. \tag{8}$$

The basic property to be used in § 3 is

$$\begin{bmatrix} \partial/\partial x_0 \\ (1/2r)X_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \\ \partial/\partial x_4 \\ \partial/\partial x_5 \\ (1/2r)X_6 \\ (1/2r)X_7 \end{bmatrix} = (1/2r)\eta A(u)\varepsilon\eta \begin{bmatrix} -\partial/\partial u_0 \\ \partial/\partial u_1 \\ \partial/\partial u_2 \\ \partial/\partial u_3 \\ \partial/\partial u_4 \\ \partial/\partial u_5 \\ \partial/\partial u_6 \\ \partial/\partial u_7 \end{bmatrix} \tag{9}$$

where the vector fields X_1, X_6 and X_7 associated to the 1-forms ω_1, ω_6 and ω_7, respectively, are

$$X_1 = c_1 u_1 \partial/\partial u_0 + u_0 \partial/\partial u_1 + c_1 u_3 \partial/\partial u_2 + u_2 \partial/\partial u_3 + c_1 u_5 \partial/\partial u_4 + u_4 \partial/\partial u_5 + c_1 u_7 \partial/\partial u_6 + u_6 \partial/\partial u_7$$

$$X_6 = -c_2 c_3 u_6 \partial/\partial u_0 + c_2 c_3 u_7 \partial/\partial u_1 + c_3 u_4 \partial/\partial u_2 - c_3 u_5 \partial/\partial u_3 - c_2 u_2 \partial/\partial u_4 + c_2 u_3 \partial/\partial u_5 + u_0 \partial/\partial u_6 - u_1 \partial/\partial u_7 \tag{10}$$

$$X_7 = c_1 c_2 c_3 u_7 \partial/\partial u_0 - c_2 c_3 u_6 \partial/\partial u_1 - c_1 c_3 u_5 \partial/\partial u_2 + c_3 u_4 \partial/\partial u_3 + c_1 c_2 u_3 \partial/\partial u_4 - c_2 u_2 \partial/\partial u_5 - c_1 u_1 \partial/\partial u_6 + u_0 \partial/\partial u_7.$$

The operators X_1, X_6 and X_7 vanish when acting on functions $\psi(x_0, x_2, x_3, x_4, x_5)$ of class $C^1(R^5)$ and satisfy the commutation relations

$$[X_1, X_6] = -2X_7 \qquad [X_6, X_7] = -2c_2c_3X_1 \qquad [X_7, X_1] = 2c_1X_6. \tag{11}$$

They therefore generate the Lie algebra su(2) or su(1, 1) according to whether $(c_1, c_2, c_3) = (-1, \pm1, \pm1)$ or $(c_1, c_2, c_3) \neq (-1, \pm1, \pm1)$. Note that in view of (10), X_1, X_6 and X_7 are defined in the Lie algebra sp(16, R).

Following the geometrical analysis developed by Lambert and Kibler (1988), and adapting it to the anti-involution j_7 inherent to the present work, the Hurwitz transformations characterised by equations (1)-(11) may be classified (up to homeomorphisms) into three types.

Type (c′). For $(c_1, c_2, c_3) = (-1, -1, -1)$, the map $R^8 \to R^5$ corresponds to the well known Hopf fibration on spheres $S^7 \to S^4$ of *compact* fibre S^3.

Type (c″). For $(c_1, c_2, c_3) = (-1, 1, 1)$, the map $R^8 \to R^+ \times R^4 \subset R^5$ corresponds to a fibration on hyperboloids, namely $R^4 \times S^3 \to R^4$ of *compact* fibre S^3.

Type (nc). For $(c_1, c_2, c_3) \neq (-1, \pm1, \pm1)$, the map $R^8 \to R^5$ corresponds to another fibration on hyperboloids, namely $R^4 \times S^3 \to R^2 \times S^2$ of *non-compact* fibre $R^2 \times S^1$.

We shall see in § 3 that there are two types of Lie algebras under constraints that we can associate to the latter three types of Hurwitz transformations.

2.2. The case 2m = 4

This case is especially simple to present since it can be obtained from the case $2m = 8$ by omitting everything involving c_3, (u_4, u_5, u_6, u_7), (x_4, x_5), (ω_6, ω_7) and (X_6, X_7). This yields the $R^4 \to R^3$ map defined by

$$x_0 = u_0^2 - c_1 u_1^2 + c_2 u_2^2 - c_1 c_2 u_3^2 \qquad x_2 = 2(-u_0 u_2 + c_1 u_1 u_3) \qquad x_3 = 2(-u_0 u_3 + u_1 u_2) \tag{12}$$

and subjected to the constraint

$$\omega_1 = 2(-u_1 \mathrm{d}u_0 + u_0 \mathrm{d}u_1 + c_2 u_3 \mathrm{d}u_2 - c_2 u_2 \mathrm{d}u_3) = 0. \tag{13}$$

In this case there is only one vector field, namely

$$X_1 = c_1 u_1 \partial/\partial u_0 + u_0 \partial/\partial u_1 + c_1 u_3 \partial/\partial u_2 + u_2 \partial/\partial u_3 \tag{14}$$

which belongs to the Lie algebra sp(8, R) with the property that $X_1 \psi(x_0, x_2, x_3) = 0$ for ψ in $C^1(R^3)$. The operator X_1 generates the subalgebra so(2) for $(c_1, c_2) = (-1, \pm1)$ and so(1, 1) for $(c_1, c_2) = (1, \pm1)$.

We thus get a transformation which coincides with the right Hurwitz transformation $\mathscr{H}_{\mathrm{R}}^{(1)}$ associated to the anti-involution j_1 of $A(c_1, c_2)$, see Lambert and Kibler (1988). The special situation where $c_1 = c_2 = -1$ leads to the transformation worked out by Kustaanheimo and Stiefel (1965). The transformation recently introduced by Iwai (1985) is obtained by taking $c_1 = -c_2 = -1$.

Here again, we have three types of Hurwitz transformations which will give two types of Lie algebras under constraints. We extract from the work of Lambert and Kibler (1988) the following classification that may be readily understood as a restriction of the corresponding one for $2m = 8$.

Type (c′). For $(c_1, c_2) = (-1, -1)$, the map $R^4 \to R^3$ corresponds to the famous Hopf fibration on spheres $S^3 \to S^2$ of *compact* fibre S^1.

Type (c″). For $(c_1, c_2) = (-1, 1)$, the map $R^4 \to R^+ \times R^2 \subset R^3$ corresponds to a fibration on hyperboloids, namely $R^2 \times S^1 \to R^2$ of *compact* fibre S^1.

Type (nc). For $(c_1, c_2) = (1, -1)$ or $(1, 1)$, the map $R^4 \to R^3$ corresponds to another fibration on hyperboloids, namely $R^2 \times S^1 \to R \times S^1$ of *non-compact* fibre R.

2.3. *The case $2m = 2$*

This limiting case presents some specific features, with respect to the cases $2m = 4$ and 8, as can be seen in terms of Laplacian and d'Alembertian operators. Nevertheless, those points of relevance for what follows may be deduced from the case $2m = 4$ by simply suppressing the expressions with c_2, (u_2, u_3) and (x_2, x_3). We are thus left with the $R^2 \to R$ map

$$x_0 = u_0^2 - c_1 u_1^2 \tag{15}$$

accompanied by the constraint

$$\omega_1 = 2(-u_1\, \mathrm{d}u_0 + u_0\, \mathrm{d}u_1) = 0. \tag{16}$$

The corresponding vector field

$$X_1 = c_1 u_1 \partial/\partial u_0 + u_0 \partial/\partial u_1 \tag{17}$$

is defined in the Lie algebra $\mathrm{sp}(4, R) \sim \mathrm{so}(3, 2)$ and satisfies $X_1 \psi(x_0) = 0$ for ψ in $C^1(R)$. The operator X_1 generates the subalgebra so(2) for $c_1 = -1$ and so(1, 1) for $c_1 = 1$. Equations (15)-(17) correspond to the right Hurwitz transformation $\mathscr{H}_{\mathrm{R}}^{(1)}$ associated to the anti-involution $j_1 \equiv j_0$ of $A(c_1)$ (cf Lambert and Kibler 1988).

It is obvious in this case that there are only two distinct Hurwitz transformations, which will produce two types of Lie algebras under constraints in § 3. Indeed, we have the following classification.

Type (c). For $c_1 = -1$, the map $R^2 \to R^+ \subset R$ corresponds to the fibration $S^1 \to \{1\}$ of *compact* fibre S^1.

Type (nc). For $c_1 = 1$, the map $R^2 \to R$ corresponds to the fibration $R \to \{1\}$ of *non-compact* fibre R.

3. Lie algebras under constraints

The study of non-bijective canonical transformations has led us to a mathematical problem that is of independent interest and has a wider realm of applications. It can be formulated as follows.

Consider a finite-dimensional Lie algebra L and one of its proper subalgebras L_0. Let L have a faithful finite-dimensional representation on some linear space M. Consider a non-bijective mapping from M to some lower-dimensional space $\tilde{M}$ such that on $\tilde{M}$ the subalgebra L_0 is represented trivially by

$$D: L_0 \to D(L_0) = 0. \tag{18}$$

The questions that we pose are as follows.

(1) Is there a uniquely defined largest subalgebra $\tilde{L}$ of L such that $L_0 \subset \tilde{L} \subseteq L$ and having a non-faithful linear representation $D: \tilde{L} \to D(\tilde{L})$ on $\tilde{M}$ with L_0 as its kernel, i.e. satisfying equation (18)?

(2) If $\tilde{L}$ exists, how one does find it and which is the largest subalgebra L_1 of $\tilde{L}$ that is represented faithfully in the representation $D(\tilde{L})$?

We start with some general Lie algebraic considerations and answer the above questions under some restrictions on L_0 and L. We then specialise to the case of interest in the context of the Hurwitz transformations of § 2, where we have $L = sp(4m, R)$ with $2m = 2$, 4 and 8, $L_0 = \{X_1\}$ for $2m = 2$ or 4 and $L_0 = \{X_1, X_6, X_7\}$ for $2m = 8$.

As far as terminology is concerned, we call L_0 a 'constraint Lie algebra' (the constraint being (18)) and L_1 a 'Lie algebra under constraints' (the constraints being brought by (18)).

3.1. General discussion

Let us first introduce some concepts that we shall need below. Here L stands for an arbitrary Lie algebra, the Lie brackets [,] of which identify with commutators in a given linear representation.

Definition 1. The normaliser of a Lie algebra L_0 in a Lie algebra L, with $L_0 \subset L$, is defined as

$$\mathrm{nor}_L L_0 = \{Z \in L \mid [Z, L_0] \subseteq L_0\}. \tag{19}$$

Thus, $\mathrm{nor}_L L_0$ is the largest subalgebra of L in which L_0 is an ideal. Given L and L_0, $\mathrm{nor}_L L_0$ is uniquely determined.

Definition 2. The centraliser of a Lie algebra L_0 in a Lie algebra L, with $L_0 \subset L$, is defined as

$$\mathrm{cent}_L L_0 = \{Z \in L \mid [Z, L_0] = 0\}. \tag{20}$$

Clearly, the subalgebra $\mathrm{cent}_L L_0$ of L is uniquely determined once L and L_0 are given.

Directly from the definitions we see that we have

$$L_0 \subseteq \mathrm{nor}_L L_0 \qquad \mathrm{nor}_L L_0 \subseteq L \qquad \mathrm{cent}_L L_0 \subseteq \mathrm{nor}_L L_0. \tag{21}$$

Let us now turn to the problem at hand. The algebra L_0 is the kernel of the Lie algebra homomorphism $D: \tilde{L} \to D(\tilde{L})$. Then, the Lie brackets

$$[D(\tilde{L}), D(L_0)] = 0 \tag{22}$$

are compatible with those of $\tilde{L}$ only if we have

$$[\tilde{L}, L_0] \subseteq L_0. \tag{23}$$

Hence, L_0 must be an ideal in $\tilde{L}$ and consequently we must have

$$\tilde{L} \subseteq \mathrm{nor}_L\, L_0. \tag{24}$$

Let us now introduce a basis $\{X_i;\ 1 \leqslant i \leqslant n\}$ for the Lie algebra L_0 (of dimension n) and complement it to a basis $\{X_i,\ Y_\alpha;\ 1 \leqslant i \leqslant n,\ 1 \leqslant \alpha \leqslant \nu\}$ for the Lie algebra $\mathrm{nor}_L\, L_0$ (of dimension $n+\nu$). The Lie brackets for $\mathrm{nor}_L\, L_0$ in this basis are

$$[X_i, X_j] = a^k_{ij} X_k \tag{25a}$$

$$[X_i, Y_\alpha] = b^j_{i\alpha} X_j \tag{25b}$$

$$[Y_\alpha, Y_\beta] = c^\gamma_{\alpha\beta} Y_\gamma + d^i_{\alpha\beta} X_i. \tag{25c}$$

If the basis $\{Y_\alpha;\ 1 \leqslant \alpha \leqslant \nu\}$ of the factor 'algebra' $F = \mathrm{nor}_L L_0/L_0$ can be so chosen that $d^i_{\alpha\beta} = 0$ $(1 \leqslant \alpha, \beta \leqslant \nu, 1 \leqslant i \leqslant n)$, then the factor algebra F is itself a Lie algebra. Moreover, in this case we have

$$L_1 = F = \{Y_\alpha;\ 1 \leqslant \alpha \leqslant \nu\} \tag{26}$$

i.e. the factor algebra F, that can be characterised as the external normaliser of L_0 in L, is itself the Lie algebra L_1 that is represented faithfully in $D(\tilde{L})$ with $\tilde{L} = \mathrm{nor}_L\, L_0$.

Relation (25*b*) provides an outer derivation of the Lie algebra L_0 unless all structure constants $b^j_{i\alpha}$ vanish. To proceed further we restrict ourselves to constraint algebras L_0 that do not have any outer derivation. According to a theorem proven by Zassenhaus (1952) (see also Jacobson 1979) this will be the case if L_0 is a finite-dimensional semisimple Lie algebra over a field of characteristic zero. On the other hand, in the case where L_0 is Abelian, a given element X_i of L_0 will either commute with all basis elements Y_α or will be represented by a nilpotent matrix in the adjoint representation of $\tilde{L}$. We thus arrive at the following results.

Lemma 1. Let the constraint Lie algebra L_0 be a semisimple Lie algebra over a field of characteristic zero. Then, the structure constants in (25*b*) satisfy

$$b^j_{i\alpha} = 0 \qquad 1 \leqslant i, j \leqslant n;\ 1 \leqslant \alpha \leqslant \nu \tag{27}$$

and we have

$$\mathrm{nor}_L\, L_0 = L_0 (+) \mathrm{cent}_L\, L_0 \tag{28}$$

where (+) denotes the direct sum of vector spaces.

Proof. Equation (27) follows from the fact that a semisimple Lie algebra has no outer derivation. The result (28) is a consequence of (27) and the fact that a semisimple Lie algebra does not have a centre, hence the condition $[X, L_0] = 0$ implies that X does not belong to L_0.

Lemma 2. Let L_0 be a subalgebra of a Cartan subalgebra of a finite-dimensional semisimple Lie algebra L over a field of characteristic zero. Then, the structure constants in (25*b*) satisfy

$$b^j_{i\alpha} = 0 \qquad 1 \leqslant i, j \leqslant n;\ 1 \leqslant \alpha \leqslant \nu \tag{29}$$

and we have

$$\mathrm{nor}_L\, L_0 = \mathrm{cent}_L\, L_0. \tag{30}$$

Proof. A Cartan subalgebra of a semisimple Lie algebra L consists entirely of elements that are represented by simultaneously diagonalisable matrices in the adjoint representation of L, at least after a field extension. A set of such matrices does not contain any nilpotent matrix. The algebra L_0 has no outer derivation so that $b^j_{i\alpha}=0$ in (25*b*). Since L_0 is Abelian, we have $a^k_{ij}=0$ in (25*a*) and the result (30) follows.

We now turn to our main results on Lie algebras under constraints.

Theorem 1. Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let L_0 be a semisimple proper subalgebra of L. The largest subalgebra $\tilde{L}$ of L that has a linear representation $D(\tilde{L})$ with L_0 as its kernel is the normaliser

$$\tilde{L}=\mathrm{nor}_L\, L_0=L_0\oplus \mathrm{cent}_L\, L_0. \tag{31a}$$

The largest subalgebra L_1 of $\tilde{L}$ that can be represented faithfully in $D(\tilde{L})$ is the centraliser

$$L_1=\mathrm{cent}_L\, L_0=\mathrm{nor}_L\, L_0/L_0. \tag{31b}$$

Proof. From lemma 1 we already know that $\mathrm{nor}_L\, L_0$ is the direct sum of the two disjoint vector spaces L_0 and $\mathrm{cent}_L\, L_0$ and that we have $b^j_{i\alpha}=0$ in (25*b*). Since $\mathrm{cent}_L\, L_0$ is a Lie algebra and X_i $(1\leqslant i\leqslant n)$ does not belong to $\mathrm{cent}_L\, L_0$, we must have $d^i_{\alpha\beta}=0$ in (25*c*) and we obtain (30). Thus, $\tilde{L}=\mathrm{nor}_L\, L_0$ is a direct sum of Lie algebras and setting $D(L_0)=0$ is consistent with representing $L_1=\mathrm{cent}_L\, L_0$ faithfully.

Theorem 2. Let L be a classical Lie algebra over the field R having an even-dimensional self-representation, i.e. a real form of A_{2N-3}, C_N or $D_N(N=2,3,\ldots)$ in Cartan's notations. Let L_0 be a one-dimensional subalgebra of a Cartan subalgebra of L, namely one of the 'diagonal subalgebras' $[\mathrm{o}(2)\oplus\mathrm{o}(2)\oplus\ldots\oplus\mathrm{o}(2)]_d$ or $[\mathrm{o}(1,1)\oplus\mathrm{o}(1,1)\oplus\ldots\oplus\mathrm{o}(1,1)]_d$. Then, the largest subalgebra $\tilde{L}$ of L that has a non-faithful representation $D(\tilde{L})$ with L_0 as its kernel is uniquely determined to be

$$\tilde{L}=\mathrm{nor}_L\, L_0=\mathrm{cent}_L\, L_0. \tag{32a}$$

The largest subalgebra L_1 of $\tilde{L}$ that can be represented faithfully in $D(\tilde{L})$ is the factor algebra

$$L_1=\mathrm{cent}_L\, L_0/L_0 \tag{32b}$$

which in this case is itself a Lie algebra.

Proof. From lemma 2 we already have $\mathrm{nor}_L\, L_0=\mathrm{cent}_L\, L_0$. We must show that under the conditions of the theorem we have

$$\mathrm{cent}_L\, L_0=L_0\oplus L_1. \tag{33}$$

By hypothesis we have $n=1$ and therefore the Lie brackets (25*a*, *b*, *c*) reduce to

$$[X_1, Y_\alpha]=0 \qquad [Y_\alpha, Y_\beta]=c^\gamma_{\alpha\beta}Y_\gamma+d^1_{\alpha\beta}X_1. \tag{34}$$

Equations (34) describe a central extension of the Lie algebra $\{Y_\alpha;\ 1\leqslant\alpha\leqslant\nu\}$ and we must show that this extension is trivial, i.e. $d^1_{\alpha\beta}=0$ $(1\leqslant\alpha,\beta\leqslant\nu)$.

Consider first the non-compact case $L_0=[\mathrm{o}(1,1)\oplus\mathrm{o}(1,1)\oplus\ldots\oplus\mathrm{o}(1,1)]_d$. We can choose a realisation of the defining faithful linear representation of L in which L_0 is represented by the matrices

$$X=a\,\mathrm{diag}[I_N,-I_N] \qquad a\in R. \tag{35a}$$

A simple calculation shows that in this representation we have

$$\mathrm{nor}_{\mathrm{L}}\, \mathrm{L}_0 = \mathrm{cent}_{\mathrm{L}}\, \mathrm{L}_0 = \{\mathrm{diag}[A, B];\ A, B \in R^{N \times N}\} \tag{35b}$$

with possibly further restrictions on the matrices A and B depending on which particular classical Lie algebra L we are considering. In any case, independently of the choice of L, the derived algebra $[\mathrm{cent}_{\mathrm{L}}\, \mathrm{L}_0, \mathrm{cent}_{\mathrm{L}}\, \mathrm{L}_0]$ of $\mathrm{cent}_{\mathrm{L}}\, \mathrm{L}_0$ is represented by matrices of the form diag$[A, B]$ with Tr $A =$ Tr $B = 0$. Hence, $\mathrm{L}_0 \not\subset [\mathrm{cent}_{\mathrm{L}}\, \mathrm{L}_0, \mathrm{cent}_{\mathrm{L}}\, \mathrm{L}_0]$ and we obtain $d^1_{\alpha\beta} = 0$ in (34) so that (33) holds.

Now consider the compact case $\mathrm{L}_0 = [\mathrm{o}(2) \oplus \mathrm{o}(2) \oplus \ldots \oplus \mathrm{o}(2)]_d$. In an appropriate realisation of the definining representation of L we have L_0 represented by the matrices

$$X = b\ \mathrm{diag}[J, J, \ldots, J] \qquad b \in R \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{36a}$$

where the matrix J occurs N times along the diagonal of X. We obtain

$$\mathrm{nor}_{\mathrm{L}}\, \mathrm{L}_0 = \mathrm{cent}_{\mathrm{L}}\, \mathrm{L}_0 = \left\{ \begin{bmatrix} X_{11} & \ldots & X_{1N} \\ \vdots & \ldots & \vdots \\ X_{N1} & \ldots & X_{NN} \end{bmatrix};\ X_{ij} = \begin{bmatrix} a_{ij} & b_{ij} \\ -b_{ij} & a_{ij} \end{bmatrix};\ a_{ij}, b_{ij} \in R;\ 1 \leqslant i, j \leqslant N \right\} \tag{36b}$$

for the normaliser (and centraliser) of L_0 in $\Delta = R^{2N \times 2N}$. Since (36*b*) provides a real representation of gl(N, C), we have $\mathrm{nor}_\Delta \mathrm{L}_0 = \mathrm{gl}(N, C)$. The normaliser of L_0 in L will be a subalgebra of gl(N, C), obtained by imposing the appropriate involution condition, reducing Δ to L. In any case, L_0 is not contained in the derived algebra sl(N, C) of gl(N, C) and still less in that of any subalgebra of gl(N, C). We again conclude that $d^1_{\alpha\beta} = 0$ in (34) so that we obtain (33).

Finding the maximal subalgebra L_1 of L that is represented faithfully when L_0 is represented trivially is thus a simple task of linear algebra and boils down, in the cases of relevance in § 3.2, to constructing the set of elements commuting elementwise with the elements of L_0. The Lie algebra L_1 coincides with what Kibler and Négadi (1983a, b, 1984a) refer to as a *Lie algebra under constraints.* In their terminology, L_1 is isomorphic to the algebra L subjected to the constraints

$$X_i = 0 \qquad 1 \leqslant i \leqslant n \tag{37}$$

and may thus be thought of as the Lie algebra surviving when the constraints (37) are introduced inside L.

3.2. Application to Hurwitz transformations

Returning to the non-bijective quadratic transformations described in § 2, we identify L as sp$(4m, R)$ with $2m = 2$, 4 or 8. The basic problem is for $\mathrm{L} = \mathrm{sp}(16, R)$ and $\mathrm{L}_0 = \mathrm{so}(3) \sim \mathrm{su}(2)$ or $\mathrm{so}(2, 1) \sim \mathrm{su}(1, 1)$ and corresponds to $2m = 8$. The two remaining problems concern $\mathrm{L} = \mathrm{sp}(8, R)$ for $2m = 4$ and $\mathrm{L} = \mathrm{sp}(4, R)$ for $2m = 2$ and both correspond to $\mathrm{L}_0 = \mathrm{so}(2)$ or so$(1, 1)$. The problems for $2m = 4$ and 2 can be solved at the same time as the problem for $2m = 8$ by adapting the restriction process of § 2.

We realise the algebra sp$(4m, R)$ by matrices X of $R^{4m \times 4m}$ satisfying

$$XK + K\tilde{X} = 0 \qquad \text{with} \qquad K = \begin{bmatrix} 0 & I_{2m} \\ -I_{2m} & 0 \end{bmatrix} \tag{38}$$

so that we have

$$X=\begin{bmatrix} A & B \\ C & -\tilde{A} \end{bmatrix} \qquad A\in R^{2m\times 2m};\; B=\tilde{B}\in R^{2m\times 2m};\; C=\tilde{C}\in R^{2m\times 2m} \tag{39}$$

(see Moshinsky and Winternitz (1980) for details). The matrix X depends on $2m(4m+1)$ parameters as it must. The Lie algebra $\mathrm{sp}(4m, R)$ is, on the one hand, realised by the matrices (39) and, on the other, by the bilinear forms

$$\alpha_{ij}=\partial_i u_j+u_j\partial_i \qquad \beta_{ij}=\partial_i\partial_j \qquad \gamma_{ij}=u_iu_j. \tag{40}$$

The representatives of the operators α_{ij}, β_{ij} and γ_{ij} in terms of matrices X may be obtained according to a simple prescription (Moshinsky and Winternitz 1980).

For $2m=2$ and 4, L_0 ($=\mathrm{so}(2)$ or $\mathrm{so}(1,1)$) is spanned by X_1 of (17) and (14), respectively. For $2m=8$, L_0 ($=\mathrm{so}(3)$ or $\mathrm{so}(2,1)$) is spanned by the three operators X_1, X_6 and X_7 of (10). It is easy to represent the constraint operators X_1, X_6 and X_7 for $\mathrm{sp}(16, R)$ in terms of matrices X of equation (39) with $2m=8$ by applying the above-mentioned prescription. The representative matrix of X_1 so obtained may serve to generate the matrices that represent the constraint operators X_1 for $\mathrm{sp}(8, R)$ and $\mathrm{sp}(4, R)$ by means of a subduction process which parallels the restriction process described in § 2 for the coordinate transformations.

It is then a simple matter of calculation to find the centraliser of $\{X_1, X_6, X_7\}$ in $\mathrm{sp}(16, R)$. It is sufficient to search for the general matrix X which commutes with the representative matrices of the operators $X_1(c_1)$ and $X_6(c_2, c_3)$ corresponding to the case $2m=8$. (The representative matrix of the operator $X_7(c_1, c_2, c_3)$ does not need to be considered since it imposes no further restriction.) This has been done in a brute force way by using the algebraic and symbolic programming system REDUCE. As a net result, the general representative matrix $X(c_1, c_2, c_3)$ of the centraliser of $\{X_1(c_1), X_6(c_2, c_3), X_7(c_1, c_2, c_3)\}$ in $\mathrm{sp}(16, R)$ is given by equation (39) with

$$A=\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18}\\
c_1a_{12} & a_{11} & c_1a_{14} & a_{13} & c_1a_{16} & a_{15} & c_1a_{18} & a_{17}\\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38}\\
c_1a_{32} & a_{31} & c_1a_{34} & a_{33} & c_1a_{36} & a_{35} & c_1a_{38} & a_{37}\\
c_3a_{37} & -c_3a_{38} & -c_2c_3a_{35} & c_2c_3a_{36} & a_{33} & -a_{34} & -c_2a_{31} & c_2a_{32}\\
-c_1c_3a_{38} & c_3a_{37} & c_1c_2c_3a_{36} & -c_2c_3a_{35} & -c_1a_{34} & a_{33} & c_1c_2a_{32} & -c_2a_{31}\\
-c_2c_3a_{17} & c_2c_3a_{18} & c_3a_{15} & -c_3a_{16} & -c_2a_{13} & c_2a_{14} & a_{11} & -a_{12}\\
c_1c_2c_3a_{18} & -c_2c_3a_{17} & -c_1c_3a_{16} & c_3a_{15} & c_1c_2a_{14} & -c_2a_{13} & -c_1a_{12} & a_{11}
\end{bmatrix} \tag{41a}$$

$$B=\begin{bmatrix}
b_{11} & 0 & b_{13} & b_{14} & b_{15} & b_{16} & 0 & 0\\
0 & -c_1b_{11} & -b_{14} & -c_1b_{13} & -b_{16} & -c_1b_{15} & 0 & 0\\
b_{13} & -b_{14} & b_{33} & 0 & 0 & 0 & c_2b_{15} & c_2b_{16}\\
b_{14} & -c_1b_{13} & 0 & -c_1b_{33} & 0 & 0 & -c_2b_{16} & -c_1c_2b_{15}\\
b_{15} & -b_{16} & 0 & 0 & c_2c_3b_{33} & 0 & -c_3b_{13} & -c_3b_{14}\\
b_{16} & -c_1b_{15} & 0 & 0 & 0 & -c_1c_2c_3b_{33} & c_3b_{14} & c_1c_3b_{13}\\
0 & 0 & c_2b_{15} & -c_2b_{16} & -c_3b_{13} & c_3b_{14} & c_2c_3b_{11} & 0\\
0 & 0 & c_2b_{16} & -c_1c_2b_{15} & -c_3b_{14} & c_1c_3b_{13} & 0 & -c_1c_2c_3b_{11}
\end{bmatrix} \tag{41b}$$

and

$$C=\text{the same as } B \text{ with } b_{ij}\rightarrow c_{ij}. \tag{41c}$$

From the matrix $X(c_1, c_2, c_3)$ so obtained, we can perform the calculation of the rank and dimension of the Lie algebra under constraints L_1, as well as the dimension of the maximal compact subalgebra of L_1 in each of the cases $\mathrm{L}=\mathrm{sp}(4m, R)$ with $2m=8$, 4 and 2. This makes it possible to identify L_1 in the following way. In the case $2m=2$ or 4, we find that $\mathrm{cent_L}\{X_1(c_1)\}$ is a Lie algebra of dimension $4m^2$ and rank $2m$ with a maximal compact subalgebra of dimension $2m^2$ for $c_1=-1$ and $m(2m-1)$ for $c_1=1$. Therefore, in the cases $2m=2$ and 4, we have $\mathrm{cent_L}\{X_1(c_1)\}=\mathrm{u}(m, m)$ or $\mathrm{gl}(2m, R)$ for $c_1=-1$ or 1, respectively. Consequently, $\mathrm{L}_1=\mathrm{cent_L}\{X_1(c_1)\}/\{X_1(c_1)\}$ is identified as $\mathrm{su}(m, m)$ for $c_1=-1$ and $\mathrm{sl}(2m, R)$ for $c_1=1$. In the case $2m=8$, we find that $\mathrm{cent_L}\{X_1(c_1), X_6(c_2, c_3), X_7(c_1, c_2, c_3)\}$ is a Lie algebra of dimension 28, of rank 4 and of character (i.e. the number of non-compact generators minus the number of compact generators) -4 for $(c_1, c_2, c_3)=(-1, \pm1, \pm1)$ and $+4$ for $(c_1, c_2, c_3)\neq(-1, \pm1, \pm1)$. Consequently, $\mathrm{L}_1=\mathrm{cent_L}\{X_1(c_1), X_6(c_2, c_3), X_7(c_1, c_2, c_3)\}$ is identified as $\mathrm{so}^*(8)$ for $(c_1, c_2, c_3)=(-1, \pm1, \pm1)$ and $\mathrm{so}(4, 4)$ for $(c_1, c_2, c_3)\neq(-1, \pm1, \pm1)$. The results for $2m=8$, 4 and 2 can be summed up and further documented as follows.

The case $2m=8$, $\mathrm{L}=\mathrm{sp}(16, R)$:

(c) $\mathrm{L}_0=\mathrm{so}(3)$ and $\mathrm{L}_1=\mathrm{so}^*(8)\sim\mathrm{so}(6, 2)$ for $(c_1, c_2, c_3)=(-1, -1, -1)$ or $(-1, 1, 1)$

(nc) $\mathrm{L}_0=\mathrm{so}(2, 1)$ and $\mathrm{L}_1=\mathrm{so}(4, 4)$ for $(c_1, c_2, c_3)\neq(-1, \pm1, \pm1)$.

The case $2m=4$, $\mathrm{L}=\mathrm{sp}(8, R)$:

(c) $\mathrm{L}_0=\mathrm{so}(2)$ and $\mathrm{L}_1=\mathrm{su}(2, 2)\sim\mathrm{so}(4, 2)$ for $(c_1, c_2)=(-1, -1)$ or $(-1, 1)$

(nc) $\mathrm{L}_0=\mathrm{so}(1, 1)$ and $\mathrm{L}_1=\mathrm{sl}(4, R)\sim\mathrm{so}(3, 3)$ for $(c_1, c_2)=(1, -1)$ or $(1, 1)$.

The case $2m=2$, $\mathrm{L}=\mathrm{sp}(4, R)$:

(c) $\mathrm{L}_0=\mathrm{so}(2)$ and $\mathrm{L}_1=\mathrm{su}(1, 1)\sim\mathrm{so}(2, 1)$ for $c_1=-1$

(nc) $\mathrm{L}_0=\mathrm{so}(1, 1)$ and $\mathrm{L}_1=\mathrm{sl}(2, R)\sim\mathrm{so}(2, 1)$ for $c_1=1$.

It is to be mentioned that the result (c) for $2m=4$ agrees with the one derived by Kibler and Négadi (1983a, b, 1984a) in the frame of a study of the hydrogen oscillator connection based on a bosonisation of the Pauli equations for the hydrogen atom.

We note the important result that, in each of the cases (n) and (nc), there is a correspondence between the types of Lie algebras under constraints and the types of fibres described in § 2. More precisely, the cases (c) correspond to *compact* fibres and the cases (nc) to *non-compact* fibres.

4. Symplectic Lie algebras under constraints

The resuls of § 3.2 can be generalised to arbitrary symplectic Lie algebras L and various constraint Lie algebras L_0. Indeed, the results obtained in § 3.2 may be derived in an alternative and more rational manner that points to further generalisations.

We shall first deal with two cases where L_0 is a one-dimensional constraint algebra and shall thus apply theorem 2. We shall then turn to two cases where L_0 is a simple Lie algebra and shall thus apply theorem 1.

4.1. The case $L_0=[o(1, 1)\oplus o(1, ,1)\oplus\ldots\oplus o(1, 1)]_d$ and $L=sp(2N, R)$

We realise the non-compact diagonal algebra L_0 by matrices of the type (39) with $2m\to N$ and

$$A_0=aI_N \qquad a\in R \qquad B_0=C_0=0. \tag{42}$$

We immediately obtain

$$\mathrm{nor_L}\,\mathrm{L}_0=\mathrm{cent_L}\,\mathrm{L}_0=\{\mathrm{diag}[A, -\tilde{A}]\}=\mathrm{gl}(N, R)=\mathrm{L}_0\oplus\mathrm{sl}(N, R). \tag{43}$$

Thus, the Lie algebra under constraints is

$$L_1 = sl(N, R) \subset \tilde{L} = gl(N, R). \tag{44a}$$

In particular, we have

$$\begin{aligned} L_1 &= sl(2, R) \sim so(2, 1) \qquad \text{for } N = 2 \\ L_1 &= sl(4, R) \sim so(3, 3) \qquad \text{for } N = 4 \end{aligned} \tag{44b}$$

in agreement with the results of § 3.2.

4.2. *The case $L_0 = [o(2) \oplus o(2) \oplus \ldots \oplus o(2)]_d$ and $L = sp(4N, R)$*

We realise the compact diagonal algebra L_0 as in (39) with $m \to N$ and

$$A_0 = a\ \mathrm{diag}[J, J, \ldots, J] \qquad a \in R \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B_0 = C_0 = 0 \tag{45}$$

where the matrix J occurs N times along the diagonal of A_0. A simple calculation yields the centraliser of L_0 in L in the form (39) where the matrix A is an $N \times N$ matrix of elements

$$\begin{bmatrix} a^1_{ij} & a^2_{ij} \\ -a^2_{ij} & a^1_{ij} \end{bmatrix} \qquad a^k_{ij} \in R;\ 1 \leq i, j \leq N;\ 1 \leq k \leq 2 \tag{46a}$$

and where the matrices B and C are given by

$$\begin{aligned} B &= A \qquad \text{with } a^k_{ij} \to b^k_{ij} \\ C &= A \qquad \text{with } a^k_{ij} \to c^k_{ij} \end{aligned} \qquad b^2_{ii} = c^2_{ii} = 0;\ 1 \leq i \leq N. \tag{46b}$$

We thus obtain

$$\mathrm{nor}_L\, L_0 = \mathrm{cent}_L\, L_0 = u(N, N) = L_0 \oplus su(N, N) \tag{47}$$

and the Lie algebra under constraints is

$$L_1 = su(N, N) \subset \tilde{L} = u(N, N). \tag{48a}$$

In particular, we have

$$\begin{aligned} L_1 &= su(1, 1) \sim so(2, 1) \qquad \text{for } N = 1 \\ L_1 &= su(2, 2) \sim so(4, 2) \qquad \text{for } N = 2 \end{aligned} \tag{48b}$$

as in § 3.2.

4.3. *The case $L_0 = [sl(2, R) \oplus sl(2, R) \oplus \ldots \oplus sl(2, R)]_d$ and $L = sp(8N, R)$*

We realise the non-compact diagonal algebra L_0, of dimension three, by the matrices (39) with $m \to 2N$ and

$$A_0 = \mathrm{diag}[J, J, \ldots, J] \qquad J = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \qquad a, b, c \in R \qquad B_0 = C_0 = 0 \tag{49}$$

where the matrix J occurs $2N$ times along the diagonal of A_0. A straightforward calculation yields the centraliser of L_0 in $\mathrm{L}=\mathrm{sp}(8N, R)$ in the form (39) where the matrices A, B and C are given by

$$A=\begin{bmatrix} a_{11} & 0 & a_{12} & 0 & \dots & a_{1,2N} & 0 \\ 0 & a_{11} & 0 & a_{12} & \dots & 0 & a_{1,2N} \\ a_{21} & 0 & a_{22} & 0 & \dots & a_{2,2N} & 0 \\ 0 & a_{21} & 0 & a_{22} & \dots & 0 & a_{2,2N} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ a_{2N,1} & 0 & a_{2N,2} & 0 & \dots & a_{2N,2N} & 0 \\ 0 & a_{2N,1} & 0 & a_{2N,2} & \dots & 0 & a_{2N,2N} \end{bmatrix}$$

$$B=\begin{bmatrix} 0 & 0 & 0 & b_{12} & \dots & 0 & b_{1,2N} \\ 0 & 0 & -b_{12} & 0 & \dots & -b_{1,2N} & 0 \\ 0 & -b_{12} & 0 & 0 & \dots & 0 & b_{2,2N} \\ b_{12} & 0 & 0 & 0 & \dots & -b_{2,2N} & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & -b_{1,2N} & 0 & -b_{2,2N} & \dots & 0 & 0 \\ b_{1,2N} & 0 & b_{2,2N} & 0 & \dots & 0 & 0 \end{bmatrix} \tag{50}$$

$$C = B \text{ with } b_{ij} \to c_{ij} \qquad a_{ij},\, b_{ij},\, c_{ij} \in R.$$

The latter realisation shows that $\mathrm{cent}_{\mathrm{L}}\, \mathrm{L}_0$ is a simple Lie algebra of dimension $2N(4N-1)$ and rank $2N$ with a maximal compact subalgebra of dimension $2N(2N-1)$. We can thus identify the centraliser and the Lie algebra under constraints as

$$\mathrm{L}_1 = \mathrm{cent}_{\mathrm{L}}\, \mathrm{L}_0 = \mathrm{so}(2N, 2N). \tag{51}$$

For $N=2$, we recover the result $\mathrm{L}_1=\mathrm{so}(4, 4)$ of § 3.2 corresponding to $\mathrm{L}_0=\mathrm{so}(2, 1)$ and $\mathrm{L}=\mathrm{sp}(16, R)$.

4.4. The case $L_0=[o(3)\oplus o(3)\oplus \ldots \oplus o(3)]_d$ and $L=sp(8N, R)$

We realise the compact diagonal algebra L_0, of dimension three, by the matrices (39) with $m\to 2N$ and

$$A_0=\mathrm{diag}[J, J, \ldots, J] \qquad J=\begin{bmatrix} 0 & a & b & c \\ -a & 0 & -c & b \\ -b & c & 0 & -a \\ -c & -b & a & 0 \end{bmatrix} \qquad a, b, c \in R \qquad B_0=C_0=0 \tag{52}$$

where the matrix J occurs N times along the diagonal of A_0. The centraliser of L_0 in $\mathrm{L}=\mathrm{sp}(8N, R)$ can be easily calculated. It is realised by matrices of the form (39) where A, B and C are $N\times N$ matrices in which each entry is a 4×4 (real) matrix of the type

$$X_{ij}=\begin{bmatrix} x_{ij}^0 & x_{ij}^1 & x_{ij}^2 & x_{ij}^3 \\ -x_{ij}^1 & x_{ij}^0 & x_{ij}^3 & -x_{ij}^2 \\ -x_{ij}^2 & -x_{ij}^3 & x_{ij}^0 & x_{ij}^1 \\ -x_{ij}^3 & x_{ij}^2 & -x_{ij}^1 & x_{ij}^0 \end{bmatrix} \qquad x_{ij}^k \in R;\ 1\leqslant i,j\leqslant N;\ 0\leqslant k\leqslant 3. \tag{53}$$

For $X \equiv B$ and C, we have $B_{ij} = \tilde{B}_{ji}$ and $C_{ij} = \tilde{C}_{ji} (1 \leq i, j \leq N)$. We find that $\text{cent}_L L_0$ is a simple Lie algebra of dimension $2N(4N-1)$ and rank $2N$ with a maximal compact subalgebra of dimension $4N^2$. We can thus identify $\text{cent}_L L_0$ as $\text{so}^*(4N)$ so that we end up with

$$L_1 = \text{cent}_L L_0 = \text{so}^*(4N). \tag{54}$$

For $N=1$, we have $\text{so}^*(4) \sim \text{so}(3) \oplus \text{so}(2,1)$. For $N=2$, we recover the result $L_1 = \text{so}^*(8) \sim \text{so}(6,2)$ of § 3.2 corresponding to $L_0 = \text{so}(3)$ and $L = \text{sp}(16, R)$.

5. Concluding remarks

The main mathematical result of this paper can be summarised in the following manner. Consider a finite-dimensional Lie algebra L and a proper subalgebra L_0 of L. Then, the largest Lie algebra $\tilde{L}$, satisfying $L_0 \subset \tilde{L} \subseteq L$ and having a non-faithful linear representation in which L_0 is represented trivially, is the normaliser $\text{nor}_L L_0$ of L_0 in L. If the normaliser allows a decomposition

$$\text{nor}_L L_0 = L_0 \oplus L_1 \tag{55}$$

into the direct sum of L_0 and a Lie algebra L_1, then L_1 can be represented faithfully in a Lie algebra homomorphism D: $\text{nor}_L L_0 \to D(\text{nor}_L L_0)$ with L_0 as its kernel.

The condition that the decomposition (55) should hold is a restriction on the algebras L_0 and L. We have shown that equation (55) always holds for the constraint algebras L_0 and the algebras L occurring in the $R^{2m} \to R^{2m-n}$ non-bijective canonical transformations with $(2m, 2m-n) = (2,1)$, $(4,3)$ and $(8,5)$.

If the decomposition (55) does not hold, then it may be necessary to enlarge the kernel of the homomorphism for $\text{nor}_L L_0 / L_0$ to be a Lie algebra. To see this, consider the example where $L = \text{sp}(4, R)$ (realised as in (39) with $m=1$) and L_0 is the one-dimensional nilpotent Lie algebra

$$L_0 = \left\{ \begin{bmatrix} 0 & 0 & b_{11} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; b_{11} \in R \right\}. \tag{56}$$

We find that

$$\text{nor}_L L_0 = \left\{ \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ 0 & a_{22} & b_{12} & b_{22} \\ 0 & 0 & -a_{11} & 0 \\ 0 & c_{22} & -a_{12} & -a_{22} \end{bmatrix}; a_{ij}, b_{ij}, c_{ij} \in R \right\}. \tag{57}$$

In this case, $\text{nor}_L L_0$ is a Lie algebra isomorphic to the 'optical Lie algebra' opt(2, 1) (see Patera *et al* 1977, Burdet *et al* 1978). Denoting A_{ij} the element of $\text{nor}_L L_0$ obtained by setting $a_{ij} = 1$ and all other entries equal to zero in equation (57), and similarly for B_{ij} and C_{ij}, we have

$$[A_{12}, B_{12}] = 2B_{11} \in L_0 \tag{58}$$

and hence $\text{nor}_L L_0 / L_0$ is not a Lie algebra. To obtain a consistent homomorphism, we must enlarge the kernel to include

$$L_0' = \{A_{12}, B_{12}, B_{11}\}. \tag{59}$$

We then have

$$\begin{aligned} &\mathrm{nor}_{\mathrm{L}}\,\mathrm{L}_0' = \mathrm{nor}_{\mathrm{L}}\,\mathrm{L}_0 \\ &\mathrm{L}_1 \sim \mathrm{nor}_{\mathrm{L}}\,\mathrm{L}_0'/\mathrm{L}_0' \sim \{A_{11} \oplus (A_{22}, B_{22}, C_{22})\} \sim \mathrm{o}(1,1) \oplus \mathrm{sl}(2, R) \end{aligned} \tag{60}$$

and L_1 is the algebra represented faithfully.

The motivation, stressed in this article, for introducing Lie algebras under constraints comes from the study of non-bijective canonical transformations. In this respect, the mathematical results obtained here are of interest in the determination of invariance and non-invariance algebras of dynamical systems (cf Kibler and Négadi 1983a, b, 1984a, Lambert and Kibler 1988). They may also be useful in related fields as in atomic and nuclear shell theory (cf Quesne 1986) and in such nuclear models as the interacting vector boson model (cf Georgieva *et al* 1986).

A different application that suggests itself concerns symmetry reduction for partial differential equations. Thus, let L be the Lie algebra of the Lie group G of local point symmetries of a system of partial differential equations (see Olver 1986) and let L_0 be a subalgebra of L corresponding to a subgroup G_0 of G. The construction of solutions invariant under the subgroup G_0 involves a non-bijective transformation from the space of independent and dependent variables $\{x_1, \ldots, x_n, u_1, \ldots, u_N\}$ to the space of G_0 invariants $\{\xi_1, \ldots, \xi_k, w_1, \ldots, w_N\}$ $(k < n)$. The transformation involves precisely the conditions

$$X\Phi(x_1, \ldots, x_n, u_1, \ldots, u_N) = 0 \qquad X \in \mathrm{L}_0. \tag{61}$$

Hence, the Lie algebra under constraints L_1 is in this case the Lie algebra of a group G_1 leaving invariant the space $\tilde{M}$ of invariants of L_0. Either G_1 or a subgroup of G_1 will then be the symmetry group of the reduced equations.

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